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ON THE EXISTENCE OF SOLUTIONS OF FULLY COUPLED FBSDES WITH MONOTONE COEFFICIENTS AND RANDOM JUMPS

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Abstract

In this work, we prove the existence of a solution of a class of forward backward stochastic differential equations (FBSDEs) with Poisson jumps by weakening the usual Lipschitz conditions on the generator of the backward equation with jumps and the drift of the forward equation with jumps. These coefficients are monotonic but can be discontinuous and the diffusion term can be degenerated.

1. Introduction

The aim of this work consists in finding a solution of a class of FBSDE with random jumps under monotonic hypotheses on the generator of the backward equation and the drift of the forward equations. More precisely, we consider the coupled system

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$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s)dW_s + \int_0^t \int_E \beta(X_{s^-}, e)\tilde{\mu}(de, ds), \\ Y_t = \Gamma + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(de, ds). \end{cases} \tag{1}$$

Fully coupled FBSDE can be encountered in various problems: The probabilistic representation of viscosity solutions of quasilinear PDE's (see [5]), the stochastic optimal control among others. In 1999, fully coupled forward-backward stochastic differential equations and their connection with PDE have been studied intensively by Pardoux and Tang (see [17]). In 2006, Antonelli and Hamadène (see [1]) gave one existence result for coupled FBSDE under non-Lipschitz assumption. Unfortunately, most existence or uniqueness results on solutions of forward-backward stochastic differential equations need regularity assumptions. The coefficients are required to be at least continuous, which is somehow too strong in some applications.

In 2008, inspired by [1], Ouknine and Ndiaye (see [15]) gave the first result, which proves existence of a solution of a forward-backward stochastic differential equation with discontinuous coefficients and degenerate diffusion coefficient where, moreover, the terminal condition is not necessary bounded. However, there is few results about reflected forward-backward stochastic differential equation in which the solution of the BSDE stays above a given barrier.

In [16], Ouknine and Ndiaye gave an extension of [15] with the obstacle constraint. Our work can be also seen as an extension of [15] with random jumps.

2. Assumptions and Notations

Let $[0, T]$ be a fixed time interval. We will always take s in $[0, T]$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, W be a d-dimensional Brownian motion defined on this space, and a Poisson random measure μ

on $\mathbb{R}_+ \times E$, where E is a compact set of $\mathbb{R}^q$, endowed with its Borel field $\mathcal{E}$. We also assume that the Poisson random measure μ is independent of W, and has the intensity measure $\lambda(de)dt$ for some finite measure λ on $(E, \mathcal{E})$. We set $\tilde{\mu}(dt, de) = \mu(dt, de) - \lambda(de)dt$, the compensated measure associated to μ. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the augmentation of the natural filtration generated by W and μ, and by $\mathcal{P}$ the σ-algebra of predictable subsets of $\Omega \times [0, T]$.

We will work with these following spaces of processes:

• $\mathcal{S}^2$, the set of adapted and continuous processes $V = (V_t)_{0\le t\le T}$ such that

$$\|V\|^2_{\mathcal{S}^2} = \mathbb{E}\left(\sup_{0\le t\le T}|V_t|^2\right) < \infty.$$

• $\mathcal{H}^2$, the set of $\mathcal{F}_t$-progressively measurable processes Z, such that

$$\|Z\|^2_{\mathcal{H}^2} = \mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty.$$

• $\mathcal{L}^p(\tilde{\mu})$, $p \ge 1$, the set of $\mathcal{P} \otimes E$-measurable maps $U : \Omega \times [0, T] \times E \to \mathbb{R}$ such that

$$\|U\|^p_{\mathcal{L}^p(\tilde{\mu})} = \mathbb{E}\left[\int_0^t \int_E |U_t(e)|^p \lambda(de)dt\right] < \infty.$$

3. Main Result

The main result of this work is given in the next theorem.

Theorem 3.1. *Let $b : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable and bounded function such that for all $s \in [0, T]$, $b(s, ., .)$ is increasing and left continuous.*

Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a measurable and bounded function such that for all* $s \in [0, T],$ $z \in \mathbb{R}, f(s, ., ., z)$ *is increasing, left continuous, and Lipschitz with respect to z uniformly in* $x, y,$ *and* s, *i.e.,* $\exists \Lambda \in \mathbb{R}_+^*$ *such that*

$$|f(s, x, y, z) - f(s, x, y, z')| \leq \Lambda(|z - z'|), \quad s \in [0, T], \quad x, y, z, z' \in \mathbb{R}.$$

Let $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying the following conditions*:

$$|\sigma(s, x)| \leq \Lambda(1 + |x|),$$

and

$$|\sigma(s, x) - \sigma(s, x')| \leq \Lambda|x - x'|, \quad s \in [0, T], \quad x, x' \in \mathbb{R}.$$

Let $\beta : \mathbb{R} \times E \to \mathbb{R}$ *be a measurable map satisfying for some positive constants C and* k_β,

$$\sup_{e \in E} |\beta(s, x)| \leq C,$$

and

$$\sup_{e \in E} |\beta(x, e) - \beta(x', e)| \leq k_\beta |x - x'|.$$

Let Γ *be a random variable* $\mathcal{F}_T$ *-measurable and square integrable.*

Then, the following fully coupled reflected forward-backward stochastic differential equations:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s) ds + \int_0^t \sigma(s, X_s) dW_s + \int_0^t \int_E \beta(X_{s^-}, e) \widetilde{\mu}(de, ds), \\ Y_t = \Gamma + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \widetilde{\mu}(de, ds), \end{cases} \tag{2}$$

has at least one solution $(X, Y, Z, U) \in \mathcal{S}^2 \otimes \mathcal{S}^2 \otimes \mathcal{H}^2 \otimes \mathcal{L}^2(\widetilde{\mu})$.

Before proving the main result, we will give two lemmas: an approximating one for increasing coefficients, which plays an important role in its proof (see [15] for the proof) and another on the comparison of solutions of BSDEs with jumps, whose proof will be given below.

Lemma 3.2. *Let* $b : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a measurable function, bounded by* M *and such that for all* $s \in [0, T]$, $b(s, ., .)$ *increasing and left continuous.*

Then, it exists a family of measurable functions $(b_n(s, x, y), n \geq 1, s \in [0, T], x, y \in \mathbb{R})$ *such that:*

(l_1) *for all sequence* $(x_n, y_n) \uparrow (x, y)$, $(x, y) \in \mathbb{R}^2$, *we have*

$$\lim_{n \to \infty} b_n(s, x_n, y_n) = b(s, x, y);$$

(l_2) $(x, y) \mapsto b_n(s, x, y)$ *is increasing, for all* $n \geq 1$, $s \in [0, T]$;

(l_3) $n \mapsto b_n(s, x, y)$ *is increasing, for all* $x \in \mathbb{R}$, $y \in \mathbb{R}$, $s \in [0, T]$;

(l_4) $|b_n(s, x, y) - b_n(s, x', y')| \leq 2nM(|x - x'| + |y - y'|)$, *for all* $n \geq 1$, $s \in [0, T]$, $M \in \mathbb{R}_+^*$;

(l_5) $\sup_{n \geq 1} \sup_{s \in [0, T]} \sup_{x, y \in \mathbb{R}} |b_n(s, x, y)| \leq M$, *for all* $n \geq 1$, $s \in [0, T]$, $x, y \in \mathbb{R}$.

Lemma 3.3. *Consider* (Y, Z, U) *and* (Y', Z', U') *the respective solutions of the following BSDEs with jumps, which generators are globally Lipschitz:*

$$Y_t = \Gamma + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \widetilde{\mu}(de, ds),$$

$$Y_t = \Gamma' + \int_t^T f'(s, Y'_s, Z'_s) ds - \int_t^T Z'_s dW_s - \int_t^T \int_E U'_s(e) \widetilde{\mu}(de, ds).$$

Assume that $\mathbb{P}$*-a.s. for any* $t \leq T$, $f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$ *and* $\Gamma \leq \Gamma'$.

Then $\mathbb{P}$*-a.s.,* $\forall t \leq T$, $Y_t \leq Y'_t$.

Proof. Let $X = (X_t)_{t \leq T}$ be a rcll semi-martingale, then by using Tanaka's formula with the function $(x^+)^2 = (\max\{x, 0\})^2$, we get

$$(X_t^+)^2 = (X_T^+)^2 - 2\int_t^T X_{s^-}^+ dX_s - \int_t^T 1_{\{X_s > 0\}} d[X^c, X^c]_s$$

$$- \sum_{t \leq s \leq T} \{(X_s^+)^2 - (X_{s^-}^+)^2 - 2X_{s^-}^+ \Delta X_s\}.$$

Here X^c denotes the continuous martingale part of X and $\Delta X_s = X_s - X_{s^-}$.

But the function $x \in \mathbb{R} \mapsto (x^+)^2$ is convex, then

$$\{(X_s^+)^2 - (X_{s^-}^+)^2 - 2X_{s^-}^+ \Delta X_s\} \geq 0.$$

From this, we deduce that

$$(X_t^+)^2 + \int_t^T 1_{\{X_s > 0\}} d[X^c, X^c]_s \leq (X_T^+)^2 - 2\int_t^T X_{s^-}^+ dX_s.$$

Now using this formula with $Y - Y'$ yields

$$((Y_t - Y_t')^+)^2 + \int_t^T 1_{\{Y_s - Y_s' > 0\}} 0]|Z_s - Z_s'|^2 ds$$

$$\leq (X_T^+)^2 - 2\int_t^T (Y_{s^-} - Y'_{s^-})^+ d(Y_s - Y_s').$$

Since $f(t, Y_t', Z_t') \leq f'(t, Y_t', Z_t')$ and f is Lipschitz, then there exist bounded and $\mathcal{F}_t$-adapted processes $(u_s)_{s \leq T}$ and $(v_s)_{s \leq T}$ such that

$$f(s, Y_s, Z_s) = f(s, Y_s', Z_s') + u_s(Y_s - Y_s') + v_s(Z_s - Z_s').$$

Therefore, we have

$$((Y_t - Y'_t)^+)^2 + \int_t^T 1_{\{Y_s - Y'_s > 0\}} 0]|Z_s - Z'_s|^2 ds$$

$$\leq 2\int_t^T (Y_{s^-} - Y'_{s^-})^+ \{u_s(Y_s - Y'_s) + v_s(Z_s - Z'_s)\} ds$$

$$- 2\int_t^T (Y_{s^-} - Y'_{s^-})^+ (Z_s - Z'_s) dW_s.$$

Taking now expectation, using the inequality $|a \cdot b| \leq \epsilon|a|^2 + \epsilon^{-1}|b|^2 (\epsilon > 0)$, and Gronwall's one, we obtain $\mathbb{E}\left[((Y_t - Y'_t)^+)^2\right] = 0$ for any $t \leq T$. The result follows since Y and Y' are rcll.

Proof of the main result

Consider the following BSDE with jumps:

$$Y_t^0 = \Gamma + M\int_t^T ds - \int_t^T Z_s^0 dW_s - \int_t^T \int_E U_s^0(e)\tilde{\mu}(ds, de). \tag{3}$$

This equation has a unique solution satisfying $\|Y_t^0\|_{\mathcal{S}^2} < \infty$.

Let us also define S as the unique solution of the SDE with jumps

$$S_t = x + \int_0^t M ds + \int_0^t \sigma(s, S_s) dW_s + \int_0^t \int_E \beta(S_{s^-}, e)\tilde{\mu}(ds, de).$$

Step 1. We will show the existence of two increasing processes $(Y^k)_{k \geq 1}$ and $(X^k)_{k \geq 1}$ satisfying

$$\begin{cases} Y_t^k = \Gamma + \int_t^T f(s, X_s^{k-1}, Y_s^k, Z_s^k) ds - \int_t^T Z_s^k dW_s - \int_t^T \int_E U_s^k(e)\tilde{\mu}(de, ds), \\ X_t^k = x + \int_0^t b(s, X_s^k, Y_s^k) ds + \int_0^t \sigma(s, X_s^k) dW_s + \int_0^t \int_E \beta(X_{s^-}^k, e)\tilde{\mu}(de, ds). \end{cases} \tag{4}$$

For $n \geq 1$, (b_n) is the sequence defined in Lemma 3.2.

Consider the following SDE with jumps:

$$X_t^{0,n} = x + \int_0^t b_n(s, X_s^{0,n}, Y_s^0)ds + \int_0^t \sigma(s, X_s^{0,n})dW_s$$

$$+ \int_0^t \int_E \beta(X_{s^-}^{0,n}, e)\widetilde{\mu}(de, ds). \tag{5}$$

According to properties (l_4), (l_5) and assumptions on β, this equation has a unique solution. $(l_3) \Rightarrow b_n(s, x, Y_s^0) \leq b_{n+1}(s, x, Y_s^0)$. We deduce from the comparison theorem of SDEs with jumps (see [18], Corollary 3.3) that the sequence $(X^{0,n})_{n\geq 1}$ is increasing. Since $b_n(s, x, Y_s^0) \leq M$, the comparison theorem of SDEs with jumps implies again $\forall t \leq T$, $X_t^{0,n} \leq S_t$ a.s.. Therefore, $X^{0,n} \nearrow X^0$.

We will show that X^0 is a solution of the SDE with jumps (5). Since $X_s^{0,n} \nearrow X_s^0$, (l_1) implies that

$$\lim_{n\to\infty} b_n(s, X_s^{0,n}, Y_s^0) = b(s, X_s^0, Y_s^0).$$

The functions $b_n(s, ., .)$ are measurable and bounded. The dominated convergence theorem gives

$$\int_0^t b_n(s, X_s^{0,n}, Y_s^0)ds \to \int_0^t b(s, X_s^0, Y_s^0)ds.$$

On the other hand,

$$\mathbb{E}\left[\int_0^t [\sigma(s, X_s^{0,n}) - \sigma(s, X_s^0)]^2 ds\right] \leq K^2 \mathbb{E}\left[\int_0^t |X_s^{0,n} - X_s^0|^2 ds\right] \to 0,$$

when $n \to \infty$. From Doob's inequality, we deduce

$$\int_0^{\cdot} \sigma(s, X_s^{0,n})dW_s \to \int_0^{\cdot} \sigma(s, X_s^0)dW_s,$$

(the limit is taking in the sense of ucp's convergence).

Similarly,

$$\mathbb{E}\left[\int_0^t\int_E (\beta(X_{s^-}^{0,n}, e) - \beta(X_{s^-}^{0}, e))\tilde{\mu}(de, ds)\right]^2$$

$$\leq \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\int_E (\beta(X_{s^-}^{0,n}, e) - \beta(X_{s^-}^{0}, e))\tilde{\mu}(de, ds)\right|^2\right]$$

$$\leq 4\mathbb{E}\left[\int_0^T\int_E |\beta(X_{s^-}^{0,n}, e) - \beta(X_{s^-}^{0}, e)|^2\lambda(de)ds\right]$$

$$\leq 4k_\beta^2\mathbb{E}\left[\int_0^T\int_E |X_{s^-}^{0,n} - X_{s^-}^{0}|^2\lambda(de)ds\right] \to 0.$$

Therefore,

$$X_t^0 = x + \int_0^t b(s, X_s^0, Y_s^0)ds + \int_0^t \sigma(s, X_s^0)dW_s + \int_0^t\int_E \beta(X_{s^-}^0, e)\tilde{\mu}(de, ds).$$

Thus, the couple of processes $(X_s^0, Y_s^0)_{s\in[0,T]}$ is well defined.

Define the random function f^1 by

$$f^1(s, y, z) := f(s, X_s^0(\omega), y, z).$$

By hypothesis, the function f is measurable, bounded, increasing, and left continuous in the y variable. Then, we can construct the following sequence of functions:

$$f_n^1(s, y, z) = n\int_{y-\frac{1}{n}}^{y} f(s, X_s^0(\omega), u, z)du.$$

$(l_1), (l_4)$, and the Lipschitz's condition with respect to y and z uniformly in x provide the existence of a unique triple of processes $(Y^{1,n}, Z^{1,n}, U^{1,n}) \in \mathcal{S}^2 \otimes \mathcal{H}^2 \otimes \mathcal{L}^2(\tilde{\mu})$ satisfying

$$Y_t^{1,n} = \Gamma + \int_t^T f_n^1(s, Y_s^{1,n}, Z_s^{1,n})ds - \int_t^T Z_s^{1,n} dW_s - \int_t^T \int_E U_s^{1,n}(e)\widetilde{\mu}(de, ds). \tag{6}$$

Since the terminal value of the RBSDE with jumps (6) is independent on n and the function $n \mapsto f_n^1(s, ., .)$ is increasing, Lemma 3.3 on the comparison theorem of RBSDEs with jumps gives us

$$\forall t \leq T \quad Y_t^0 \leq Y_t^{1,n} \leq Y_t^{1,n+1}.$$

Now, let us prove the convergence of the sequences $(Y_t^{1,n})_{n\geq 0}$ and $(Z_t^{1,n})_{n\geq 0}$.

Indeed, it follows from Itô's formula that

$$|Y_t^{1,n}|^2 + \int_t^T |Z_s^{1,n}|^2 ds + \int_t^T ds \int_E (U_s^{1,n}(e))^2 \lambda(de) + \sum_{t\leq s\leq T} (\Delta_s Y_s^{1,n})^2$$

$$= \Gamma^2 + 2\int_t^T Y_s^{1,n} f_n^1(s, Y_s^{1,n}, Z_s^{1,n})ds - 2\int_t^T Y_{s_-}^{1,n} Z_s^{1,n} dW_s$$

$$- 2\int_t^T \int_E Y_{s_-}^{1,n} U_s^{1,n}(e)\widetilde{\mu}(de, ds).$$

Let us denote by N_t the local martingale

$$\int_0^t Y_{s_-}^{1,n} Z_s^{1,n} dW_s + \int_0^t \int_E Y_{s_-}^{1,n} U_s^{1,n}(e)\widetilde{\mu}(de, ds).$$

Then, we have

$$\sup_{0\leq t\leq T} |Y_t^{1,n}|^2 \leq \Gamma^2 + 2 \sup_{0\leq t\leq T} |N_T - N_t| + 2\int_t^T |Y_s^{1,n}| \cdot |f_n^1(s, Y_s^{1,n}, Z_s^{1,n})| ds. \tag{7}$$

By Burkholder-Davis-Gundy's inequality for local martingales, we know that there exists a constant ϱ such that

$$\mathbb{E}\left(\sup_{0\le t\le T}|N_t|\right) \le \varrho\mathbb{E}\left([N, N]_T^{1/2}\right).$$

A computation gives

$$\mathbb{E}\left([N, N]_T^{1/2}\right) = \mathbb{E}\left[\left(\int_0^T |Y_s^{1,n}|^2 |Z_s^{1,n}|^2 ds + \int_0^T \int_E |Y_s^{1,n}|^2 |U_s^{1,n}(e)|^2 \lambda(de) ds\right)^{1/2}\right]$$

$$\le \mathbb{E}\left[\sup_{0\le s\le T}|Y_s^{1,n}|\left(\int_0^T |Z_s^{1,n}|^2 ds + \int_0^T \int_E |U_s^{1,n}(e)|^2 \lambda(de) ds\right)^{1/2}\right]$$

$$\le \frac{\varepsilon}{2}\mathbb{E}\left[\sup_{0\le s\le T}|Y_s^{1,n}|^2\right] + \frac{1}{2\varepsilon}\mathbb{E}\left(\int_0^T |Z_s^{1,n}|^2 ds + \int_0^T \int_E |U_s^{1,n}(e)|^2 \lambda(de) ds\right),$$

for any $\varepsilon > 0$.

Using boundedness property of f_n^1, one gets

$$2|Y_s^{1,n}| \cdot |f_n^1(s, Y_s^{1,n}, Z_s^{1,n})| \le 2M|Y_s^{1,n}|.$$

By (8), we have

$$\mathbb{E}\left[\sup_{0\le s\le T}|Y_s^{1,n}|^2\right] \le \mathbb{E}\left[\Gamma^2\right] + (2MT + \varepsilon\varrho)\mathbb{E}\left[\sup_{0\le s\le T}|Y_s^{1,n}|^2\right]$$

$$+ \frac{\varrho}{\varepsilon}\mathbb{E}\left[\int_0^T |Z_s^{1,n}|^2 ds\right] + \frac{\varrho}{\varepsilon}\mathbb{E}\left[\int_0^T \int_E |U_s^{1,n}(e)|^2 \lambda(de) ds\right].$$

Finally, by choosing $(2MT + \varepsilon\varrho) < 1$, we obtain $\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{1,n}|^2\right] < \infty$.

Let $Y_t^1 = \liminf_{n\to\infty} Y_t^{1,n}$, $t \le T$. Since the sequence $(Y^{1,n})_{n\ge 0}$ is non-decreasing, then using Fatou's lemma, we have that for any $t \le T$, $Y_t^1 < \infty$, and then $\mathbb{P}$-a.s., $Y_s^{1,n} \to Y_s^1$ as $n \to \infty$. In addition, the Lebesgue's dominated convergence theorem implies that

$\mathbb{E}\left[\int_0^T |Y_s^{1,n} - Y_s^1|^2\right] ds \to 0$ as $n \to \infty$. For the sequence $(Z_t^{1,n})_{n\geq 0}$, let us apply the Itô's formula to the function $x \mapsto |x|^2$ and the difference of processes $Y_s^{1,k} - Y_s^{1,h}$ between s and T. Then

$$|Y_t^{1,k} - Y_t^{1,h}|^2 + \int_t^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds + \int_t^T \int_E |U_s^{1,k}(e) - U_s^{1,h}(e)|^2 \lambda(de) ds$$

$$+ \sum_{t\leq s\leq T} \Delta_s (Y^{1,k} - Y^{1,h})^2 = 2\int_t^T (Y_s^{1,k} - Y_s^{1,h})[f_1^k(s, Y_s^{1,k}, Z_s^{1,k})$$

$$- f_1^h(s, Y_s^{1,h}, Z_s^{1,h})] ds - 2\int_t^T (Y_{s_-}^{1,k} - Y_{s_-}^{1,h})(Z_s^{1,k} - Z_s^{1,h}) dW_s$$

$$- 2\int_t^T \int_E (Y_{s_-}^{1,k} - Y_{s_-}^{1,h})(U_s^{1,k}(e) - U_s^{1,h}(e))\widetilde{\mu}(ds, de). \tag{8}$$

Taking the expectation in each member of (8) and taking into account that the stochastic integrals $(\int_0^t (Y_{s_-}^{1,k} - Y_{s_-}^{1,h})(Z_s^{1,k} - Z_s^{1,h}) dW_s)_{0\leq t\leq T}$ and $(\int_0^t \int_E (Y_{s_-}^{1,k} - Y_{s_-}^{1,h})(U_s^{1,k}(e) - U_s^{1,h}(e))\widetilde{\mu}(ds, de))_{0\leq t\leq T}$ are martingales leads to

$$\mathbb{E}\left[|Y_t^{1,k} - Y_t^{1,h}|^2 + \int_t^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds + \int_t^T \int_E |U_s^{1,k}(e) - U_s^{1,h}(e)|^2 \lambda(de) ds\right]$$

$$\leq 2\mathbb{E}\left[\int_t^T (Y_s^{1,k} - Y_s^{1,h})[f_1^k(s, Y_s^{1,k}, Z_s^{1,k}) - f_1^h(s, Y_s^{1,h}, Z_s^{1,h})] ds\right].$$

Hölder inequality, we have

$$\mathbb{E}(\int_0^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds)$$

$$\leq \mathbb{E}\Big[|Y_s^{1,k} - Y_s^{1,h}|^2 + \int_0^T |Z_s^{1,k} - Z_s^{1,h}|^2 ds$$

$$+ \int_t^T \int_E |U_s^{1,k}(e) - U_s^{1,h}(e)|^2 \lambda(de)ds\Bigg]$$

$$\leq K_1[\mathbb{E}(\int_0^T [f_1^k(s, Y_s^{1,k}, Z_s^{1,k}) - f_1^h(s, Y_s^{1,h}, Z_s^{1,h})]^2 ds)]^{\frac{1}{2}}$$

$$\times [\mathbb{E}(\int_0^T (Y_s^{1,k} - Y_s^{1,h})^2 ds)]^{\frac{1}{2}} \to 0,$$

because the functions f_1^k are bounded and the sequence $(Y_t^{1,k})_{k\geq 1}$ is convergent.

So, the sequence $(Z_t^{1,n})_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}^2$. Thus, it converges to a limit $Z^1 \in \mathcal{H}^2$. Here K_1 is a constant.

Similarly, the sequence $(U_t^{1,n})_{n\geq 0}$ is a Cauchy sequence in $\mathcal{L}^2(\tilde{\mu})$. Thus, it converges to a limit $U^1 \in \mathcal{L}^2(\tilde{\mu})$.

On the other hand, $(Y_s^{1,n}, Z_s^{1,n}) \to (Y_s^1, Z_s^1)$ and because of (1_1), we have

$$f_n^1(s, Y_s^{1,n}, Z_s^{1,n}) \to f^1(s, Y_s^1, Z_s^1) = f(s, X_s^0, Y_s^1, Z_s^1).$$

Since the functions f_n^1 are measurable and bounded, the dominated convergence theorem implies

$$\lim_{n\to\infty} \int_t^T f_n^1(s, X_s^0, Y_s^1, Z_s^1)ds = \int_t^T f(s, X_s^0, Y_s^1, Z_s^1)ds.$$

We also have

$$\int_t^T Z_s^{1,n} dW_s \to \int_t^T Z_s^1 dW_s.$$

Moreover, $\int_t^T \int_E U_s^{1,n}(e)\widetilde{\mu}(de, ds) \to \int_t^T \int_E U_s^1(e)\widetilde{\mu}(de, ds)$ in the sense that

$$\mathbb{E}\left\{\left[\int_t^T \int_E (U_s^{1,n}(e) - U_s^1(e))\widetilde{\mu}(de, ds)\right]^2\right\} = \mathbb{E}\left[\int_t^T \int_E |U_s^{1,n}(e) - U_s^1(e)|^2 \lambda(de)ds\right] \to 0,$$

as $n \to \infty$.

Finally, we obtain a triple $(Y_t^1, Z_t^1, U_t^1)_{0 \le t \le T}$ satisfying the following equation:

$$Y_t^1 = \Gamma + \int_t^T f(s, X_s^0, Y_s^1, Z_s^1)ds - \int_t^T Z_s^1 dW_s - \int_t^T \int_E U_s^1(e)\widetilde{\mu}(de, ds).$$

Taking the limit, we also have $\forall t \le T$, $Y_t^0 \le Y_t^1$, and $\|Y_t^1\|_{\mathcal{S}^2} < \infty$.

Next, consider the forward component linked with Y^1,

$$X_t^{1,n} = x + \int_0^t b_n(s, X_s^{1,n}, Y_s^{1,n})ds + \int_0^t \sigma(s, X_s^{1,n})dW_s$$

$$+ \int_0^t \int_E \beta(X_{s^-}^{1,n}, e)\widetilde{\mu}(de, ds). \tag{9}$$

Since $Y^0 \le Y^1$ and $(b_n(s, ., .))_{n \ge 0}$ is increasing in space and with respect to n, we have

$$b_n(s, x, Y_s^0) \le b_n(s, x, Y_s^1) \le b_{n+1}(s, x, Y_s^1),$$

we also have through the comparison theorem of SDE with jumps that

$$\forall t \le T, \quad X_t^{0,n} \le X_t^{1,n} \le X_t^{1,n+1}. \tag{10}$$

Repeating what we have done on the construction of X^0, we can show the existence of a process X^1 in $\mathcal{S}^2$, which is an increasing limit of the sequence $(X^{1,n})_{n \ge 0}$ and such that

$$\forall t \leq T, \quad X_t^1 = x + \int_0^t b(s, X_s^1, Y_s^1) ds + \int_0^t \sigma(s, X_s^1) dW_s$$

$$+ \int_0^t \int_E \beta(X_{s^-}^1, e) \tilde{\mu}(de, ds).$$

Taking the limit in (10), one gets $\forall t \leq T, \quad X_t^0 \leq X_t^1$.

Having found a solution $(X^1, Y^1, Z^1, U^1) \in \mathcal{S}^2 \otimes \mathcal{S}^2 \otimes \mathcal{H}^2 \otimes \mathcal{L}^2(\tilde{\mu})$ of (1), we can proceed by induction to find the anticipated solution.

Step 2. Let us suppose that we built the sequence of solutions (X^i, Y^i, Z^i, U^i) for all $i \leq k-1$, i.e., for all $i = 1, \cdots, k-1$ and $t \leq T$,

$$\begin{cases} X_t^i = x + \int_0^t b(s, X_s^i, Y_s^i) ds + \int_0^t \sigma(s, X_s^i) dW_s + \int_0^t \int_E \beta(X_{s^-}^i, e) \tilde{\mu}(de, ds), \\ Y_t^i = \Gamma + \int_t^T f(s, X_s^{i-1}, Y_s^i, Z_s^i) ds - \int_t^T Z_s^i dW_s - \int_t^T \int_E U_s^i(e) \tilde{\mu}(de, ds). \end{cases}$$

For $t \leq T$, $X_t^{i-1} \leq X_t^i$, $Y_t^{i-1} \leq Y_t^i$, and $\|Y_t^i\|_{\mathcal{S}^2} < \infty$.

Define the random function

$$f^k(s, y, z) := f(s, X_s^{k-1}(\omega), y, z).$$

By hypothesis, $f^k(s, y, z)$ is measurable and bounded. Then, we can build the sequence of functions f_n^k satisfying (l_1), (l_2), (l_3), (l_4), and (l_5).

Now, consider the following BSDE with jumps:

$$Y_t^{k,n} = \Gamma + \int_t^T f_n^k(s, Y_s^{k,n}, Z_s^{k,n}) ds - \int_t^T Z_s^{k,n} dW_s - \int_t^T \int_E U_s^k(e) \tilde{\mu}(de, ds).$$

Since $f^k(s, y, z) = f(s, X_s^{k-1}(\omega), y, z)$, $f^{k-1}(s, y, z) = f(s, X_s^{k-2}(\omega), y, z)$, and $X_s^{k-1} \leq X_s^{k-2}$, the increase of the function f in x implies that $f^{k-1}(s, y, z) \leq f^k(s, y, z)$. What allows us to say that $f_n^{k-1}(s, y, z) \leq f_n^k(s, y, z)$, $\forall n \geq 0$. Thus, the comparison's Lemma 3.3 for BSDEs with jumps gives us

$$\forall t \leq T, \quad Y_t^{k-1, n} \leq Y_t^{k, n}. \tag{11}$$

The same calculations done with $Y_t^{1, n}$ show that

$$\sup_{n, k} \|Y^{k, n}\|_{\mathcal{S}^2} < \infty. \tag{12}$$

We deduct that from it the sequence $(Y^{k, n})_{n \geq 0}$ is convergent in $\mathcal{S}^2$ to a process denoted Y^k.

The same calculation made with $Z_t^{1, n}$ allows to say that the sequence $(Z^{k, n})_{n \geq 0}$ is convergent in $\mathcal{S}^2$ to a process denoted Z^k. Then $(Y_s^{k, n}, Z_s^{k, n}) \to (Y_s^k, Z_s^k)$ and $Y_s^{k, n} \nearrow Y_s^k$.

By virtue of (l_1), we have

$$\lim_{n \to \infty} f_n^k(s, Y_s^{k, n}, Z_s^{k, n}) = f^k(s, Y_s^k, Z_s^k) = f(s, X_s^{k-1}, Y_s^k, Z_s^k).$$

Since the functions f_n^k are measurable and bounded, the dominated convergence theorem implies

$$\lim_{n \to \infty} \int_t^T f_n^k(s, Y_s^{k, n}, Z_s^{k, n}) ds = \int_t^T f(s, X_s^{k-1}, Y_s^k, Z_s^k) ds.$$

On the other hand, $X_s^{k, n} \to X_s^k$.

Moreover, $\int_t^T \int_E U_s^{k, n}(e) \widetilde{\mu}(de, ds) \to \int_t^T \int_E U_s^k(e) \widetilde{\mu}(de, ds)$ in the sense that

$$\mathbb{E}\left\{\left[\int_t^T \int_E (U_s^{k, n}(e) - U_s^k(e)) \widetilde{\mu}(de, ds)\right]^2\right\}$$

$$= \mathbb{E}\left[\int_t^T \int_E |U_s^{k, n}(e) - U_s^k(e)|^2 \lambda(de) ds\right] \to 0,$$

as $n \to \infty$.

As in the previous step, we obtain a triple $(Y_t^k, Z_t^k, U_t^k)_{0\le t\le T}$ satisfying the following equation:

$$Y_t^k = \Gamma + \int_t^T f(s, X_s^{k-1}, Y_s^k, Z_s^k)ds - \int_t^T Z_s^k dW_s - \int_t^T \int_E U_s^k(e)\widetilde{\mu}(de, ds).$$

Taking the limit in (11) and (12), together with $\forall t \le T$, $Y_t^{k-1} \le Y_t^k$ leads to $\|Y_t^k\|_{\mathcal{S}^2} < \infty$.

The sequence $(Y^k)_k$ is increasing and bounded, it converges on one process, which we shall denote by Y_t. We need to show now that the sequence $(Z^k)_k$ is a Cauchy sequence.

Applying the Itô's formula to the function $x \mapsto |x|^2$ and to the process $Y^k - Y^h$ between t and T, we obtain

$$\begin{aligned}(Y_t^k - Y_t^h)^2 &= 2\int_t^T (Y_s^k - Y_s^h)[f_1^k(s, X_s^{k-1}, Y_s^k, Z_s^k) - f_1^h(s, X_s^{h-1}, Y_s^h, Z_s^h)]ds \\ &\quad + \sum_{t\le s\le T} \Delta_s(Y^k - Y^h)^2 - \int_t^T |Z_s^k - Z_s^h|^2 ds \\ &\quad - \int_t^T \int_E |U_s^k(e) - U_s^h(e)|^2 \lambda(de)ds \\ &\quad - 2\int_t^T (Y_{s_-}^k - Y_{s_-}^h)(Z_s^k - Z_s^h)dW_s \\ &\quad - 2\int_t^T \int_E (Y_{s_-}^k - Y_{s_-}^h)(U_s^k(e) - U_s^h(e))\widetilde{\mu}(ds, de).\end{aligned}$$

But $(\int_0^t (Y_{s_-}^k - Y_{s_-}^h)(Z_s^k - Z_s^h)dW_s)_{0\le t\le T}$ and $(\int_0^t \int_E (Y_{s_-}^k - Y_{s_-}^h)(U_s^k(e) - U_s^h(e))\widetilde{\mu}(ds, de))_{0\le t\le T}$ are martingales. As previously, by taking the expectation in each member and by using Hölder's inequality, we obtain

$$\mathbb{E}\left[\int_0^T |Z_s^k - Z_s^h|^2 ds\right]$$

$$\leq 2\left[\mathbb{E}\left(\int_0^T [f(s, X_s^{k-1}, Y_s^k, Z_s^k) - f(s, X_s^{h-1}, Y_s^h, Z_s^h)]^2 ds\right)\right]^{\frac{1}{2}}$$

$$\times\left[\mathbb{E}\left(\int_0^T (Y_s^k - Y_s^h)^2 ds\right)\right]^{\frac{1}{2}}.$$

But because $f(s, ., ., .)$ is bounded and $Y^k - Y^h \to 0,$ we have $\mathbb{E}[\int_0^T |Z_s^k - Z_s^h|^2 ds] \to 0.$ Then $(Z^k)_{k\geq 0}$ is a Cauchy sequence in $\mathcal{H}^2$ with $Z = \lim_{k\to\infty} Z^k.$

Similarly, the sequence $(U^k)_{k\geq 0}$ is a Cauchy sequence in $\mathcal{L}^2(\widetilde{\mu})$ with $U = \lim_{k\to\infty} U^k.$

Let us return to the forward component and let us consider the SDE with jumps

$$X_t^{k,n} = x + \int_0^t b_n(s, X_s^{k,n}, Y_s^k)ds + \int_0^t \sigma(s, X_s^{k,n})dW_s$$

$$+ \int_0^t \int_E \beta(X_{s^-}^{k,n}, e)\widetilde{\mu}(de, ds).$$

By repeating the same work made with $X^1,$ i.e., by changing 1 in k, we obtain the same conclusion for X^k to know

$$X_t^{k-1,n} \leq X_t^{k,n} \leq S_t, \quad X_t^{k,n} \to X_t^k,$$

$$X_t^k = x + \int_0^t b_n(s, X_s^k, Y_s^k)ds + \int_0^t \sigma(s, X_s^k)dW_s + \int_0^t \int_E \beta(X_{s^-}^k, e)\widetilde{\mu}(de, ds),$$

$$X_t^{k-1} \leq X_t^k \leq S_t.$$

The sequence $(X^k)_k$ is increasing and bounded above, then it converges in $\mathcal{H}^2$ to a process denoted X.

By the left continuity of b, we have $b(s, X_s^k, Y_s^k) \to b(s, X_s, Y_s)$ when $k \to \infty$.

Since the function $b(s, ., .)$ is measurable and bounded, the dominated convergence theorem implies

$$\int_0^t b(s, X_s^k, Y_s^k) \to \int_0^t b(s, X_s, Y_s) ds.$$

On the other hand,

$$\mathbb{E}\left[\int_0^t [\sigma(s, X_s^k) - \sigma(s, X_s)]^2 ds\right] \leq K^2 \mathbb{E}\left[\int_0^t |X_s^k - X_s|^2 ds\right] \to 0,$$

when $n \to \infty$ since $X_s^k \to X_s$. Then $\int_0^{\cdot} \sigma(s, X_s^k) dW_s \to \int_0^{\cdot} \sigma(s, X_s) dW_s$.

Moreover,

$$\mathbb{E}\left[\int_0^t \int_E (\beta(X_{s^-}^k, e) - \beta(X_{s^-}, e))\widetilde{\mu}(de, ds)\right]^2$$

$$\leq \mathbb{E}\left[\sup_{0 \leq t \leq T} \left|\int_0^t \int_E (\beta(X_{s^-}^k, e) - \beta(X_{s^-}, e))\widetilde{\mu}(de, ds)\right|^2\right]$$

$$\leq 4\mathbb{E}\left[\int_0^T \int_E |\beta(X_{s^-}^k, e) - \beta(X_{s^-}, e)|^2 \lambda(de) ds\right]$$

$$\leq 4k_\beta^2 \mathbb{E}\left[\int_0^T \int_E |X_{s^-}^k - X_{s^-}|^2 \lambda(de) ds\right] \to 0.$$

Then $\int_0^{\cdot} \int_E (\beta(X_{s^-}^k, e))\widetilde{\mu}(de, ds) \to \int_0^{\cdot} \int_E (\beta(X_{s^-}, e))\widetilde{\mu}(de, ds)$. So,

$$X_t = x + \int_0^t b(s, X_s, Y_s) ds + \int_0^t \sigma(s, X_s) dW_s + \int_0^t \int_E (\beta(X_{s^-}, e))\widetilde{\mu}(de, ds).$$

Let us show now that $\forall t \leq T, (X_t, Y_t, Z_t, U_t)_{t \leq T}$ satisfies

$$Y_t = \Gamma + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\widetilde{\mu}(de, ds). \quad (13)$$

Since $\lim_{k \to \infty} X_t^k = X, \lim_{k \to \infty} Y_t^k = Y, \lim_{k \to \infty} Z_t^k = Z,$ and f is left continuous in y and Lipschitz in z

$$f(X_s^k, Y_s^k, Z_s^k) \to f(X_s, Y_s, Z_s).$$

Moreover f is measurable and bounded, then the dominated convergence theorem implies that

$$\int_t^T f(s, X_s^k, Y_s^k, Z_s^k)ds \to \int_t^T f(s, X_s, Y_s, Z_s)ds.$$

On the other hand, $Z_s^k \to Z_s$. So $\mathbb{E}\int_0^T |Z_s^k - Z_s|^2 ds \to 0,$ leading to $\int_0^T Z_s^k dW_s \to \int_0^T Z_s dW_s$. Moreover, $\int_t^T \int_E U_s^k(e)\widetilde{\mu}(de, ds) \to \int_t^T \int_E U_s(e)$ $\widetilde{\mu}(de, ds)$ in the sense that

$$\mathbb{E}\left\{\left[\int_t^T \int_E (U_s^k(e) - U_s(e))\widetilde{\mu}(de, ds)\right]^2\right\}$$

$$= \mathbb{E}\left[\int_t^T \int_E |U_s^k(e) - U_s(e)|^2 \lambda(de)ds\right] \to 0,$$

as $n \to \infty$.

Finally, $\forall t \leq T, (X_t, Y_t, Z_t, U_t)_{t \leq T}$ clearly satisfies (13) and the proof ends.

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